

# $O(a)$ errors in 3-D $SU(N)$ Higgs theories

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## Abstract

We compute the matching conditions between lattice and continuum 3-D  $SU(N)$  Higgs theories, with both adjoint and fundamental scalars, at  $O(a)$ , except for additive corrections to masses and  $\phi^2$  insertions.

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## 1 Introduction

The study of early universe cosmology demands the use of particle physics, because Hubble expansion means that the universe began at a very large temperature and density. By looking back to time scales on order seconds and temperatures on order 1MeV, and using QED, nuclear physics, low energy weak interaction physics, and nonequilibrium kinetics, it has been possible to predict the primordial abundances of elements. It is natural to try to push this success back to shorter times and higher energies, where one finds a sequence of interesting phase transitions: the QCD phase transition, which it may soon be possible to study through heavy ion collisions; the electroweak phase transition, which may be responsible for baryogenesis; and perhaps even GUT phase transitions, which may have bearing on baryogenesis, inflation, and vacuum stability.

Our attempts to understand these phase transitions, and thermal physics at these energy scales more generally, have been hampered for two reasons. First is an uncertainty about the underlying physics, in the electroweak case and especially in the GUT case. The second is the infrared problem of interacting, bosonic thermal field theories. Even at weak coupling, if one is interested in sufficiently long wavelength phenomena, mutual interactions between light bosonic species cause ordinary perturbation theory to break down. Perturbation theory is inevitably useless in the high temperature phase of Yang-Mills Higgs theories when one considers length scales of order  $1/g^2T$ , and for shorter length scales this problem invariably enters calculations at some perturbative order, because of interactions with modes at this scale.

This problem was first noted by Linde [1], and has since been understood as arising from the essentially 3 dimensional nature of the thermodynamics of the infrared, bosonic fields. The partition function describing equal time, equilibrium thermodynamics,  $Z = \text{Tr} e^{-\beta H}$ ,

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can be written as a Euclidean path integral where “time” is periodic with period  $\beta = 1/T$ . At weak coupling, the decoupling theorem allows one to integrate out Fourier modes with nonzero wave number in this direction, the nonzero Matsubara frequency modes[2]. This leaves a three dimensional theory, which is superrenormalizable, and therefore well behaved in the ultraviolet but potentially poorly behaved in the infrared. The couplings of this theory have nonzero engineering dimension, which sets a natural scale where the perturbative expansion becomes unreliable. The physics of interest often lies in this region.

The solution to this problem is to perform the integration over the nonzero Matsubara frequencies analytically, reducing the theory to the simplest sub-theory which contains the correct problematic infrared behavior. Then one studies the 3-D theory nonperturbatively, for instance on a lattice. The lattice treatment also turns out to be particularly tractable because of the reduced dimension and the superrenormalizability.

In the case of the electroweak phase transition this program has been pushed nearly to completion. The dimensional reduction has been performed [3], and the match between the 3-D continuum theory and a 3-D lattice theory has been carried out so that they will have the same small lattice spacing  $a$  limit [4, 5]. This matching procedure can be exact because of the superrenormalizability of the 3-D theory. Several groups have studied the resulting system numerically [6, 7, 8, 9, 10]. There has also been a very general analysis of the dimensional reduction step [11], which has been applied to the supersymmetric extension of the theory as well [12].

Numerical studies of the lattice theory display quite strong  $O(a)$  systematic errors, which at first were removed by extrapolation [6]. Except for an additive correction to  $\Phi^2$  and the value of the Higgs mass parameter at the equilibrium point, all of these errors arise from one loop corrections to the lattice continuum match and can be absorbed into a renormalization of the couplings and wave functions of the lattice theory. In particular such a renormalization removes all  $O(a)$  errors in the determination of observables which describe the strength of the phase transition. The details of this  $O(a)$  improvement step, in the case of  $SU(2) \times U(1)$  Higgs theory, are worked out in [13]. (There is a numerically unimportant combinatorial error there, and two in Appendix C. We will correct them here.)

Recently there has been interest in applying the dimensional reduction program to several other theories. It may be possible to understand the thermodynamics of hot QCD above the chiral phase transition in terms of  $SU(3) +$  adjoint Higgs theory [14, 15]. Superconductivity is described by a Landau-Ginsburg model which is 3-D  $U(1) +$  Higgs theory, which also describes the Kibble mechanism for cosmic string formation. It has been the topic of recent study [16]. If the stop squark is light, then it may be necessary to study the supersymmetric Standard Model by dimensional reduction to  $SU(3) \times SU(2)$  theory with a fundamental Higgs field for each group [17]. It will also be necessary to use dimensional reduction and a numerical study of the reduced theory to understand the details of GUT phase transitions, such as the  $SU(5)$  breaking phase transition [18].

In each case it is best to perform both the dimensional reduction calculation and the lattice to continuum matching calculation before undertaking extensive numerical work. In most cases the dimensional reduction step has been performed; otherwise the generic rules derived in [11] can be used. Recently, Laine and Rajantie have computed the lattice continuum relations for scalar masses and the order parameter  $\Phi^2$ , for 3-D  $SU(N)$  gauge theory with either a fundamental or an adjoint Higgs field, at two loops [19]. This is sufficient

to ensure the lattice model has the correct small  $a$  limit. It seems profitable also to extend the  $O(a)$  matching between the lattice and continuum theories to the general case, so that measurables relating to the strength of the phase transition can be computed up to  $O(a^2)$  corrections. We do this here.

Here is an outline of the remainder of the paper. In Section 2 we will explain the general idea of the  $O(a)$  improvement, and why it is possible. Section 3 will present the coefficients of the  $O(a)$  corrections for  $SU(N) + \text{adjoint Higgs} + \text{fundamental Higgs}$ , with comments on how to remove either Higgs field. Section 4 shows what to do when the gauge group is  $SU(N) \times SU(M)$ , each with Higgs fields. The last section is the conclusion. There are also two appendices; the first estimates  $O(a^2)$  errors by computing a particularly simple and important subset of graphs, and the second gives numerical values in particular theories and step by step instructions for building  $O(a)$  improved lattice actions.

## 2 $O(a)$ improvement: the general idea

Considerable attention has been given to the problem of improving the match between lattice and continuum regulated quantum field theories. Most attention has focused on the case of 4-D theories, which are relevant for vacuum field theory. For instance, in 4-D pure  $SU(N)$  Yang-Mills theory with no fermions, the continuum action is

$$\int d^4x \frac{1}{2} \text{Tr} F_{\mu\nu} F_{\mu\nu} , \quad (1)$$

where  $F_{\mu\nu} = F_{\mu\nu}^a T^a = (i/g_c)[D_\mu, D_\nu]$ , and  $D_\mu = \partial_\mu - ig_c T^a A_\mu^a$  is the the fundamental representation covariant derivative. Here it is understood that divergences are to be removed in the  $\overline{\text{MS}}$  scheme at a renormalization point  $\mu$  and that  $g_c$  refers to the value at that renormalization point.

A corresponding lattice action is

$$\frac{2}{g_l^2 a^4} \sum_x \sum_{i < j} \text{Tr} (\mathbf{1} - P_{ij}(x)) , \quad (2)$$

where  $P_{ij}(x)$  is the product of link matrices around a square in the  $i, j$  direction and starting at the lattice site  $x$ .

What we want is a match of this lattice theory to the continuum theory such that they produce the same infrared effective theory, up to small and controlled errors. The general philosophy of the matching procedure was developed by Symanzik years ago [20]. If the theory were strictly linear then the matching procedure would be trivial; setting  $g_l^2 = g_c^2$ , we could reproduce the behavior of the continuum theory up to  $O(a^2)$  errors associated with nonrenormalizable operators. However, the nonlinearity of the theory makes the infrared behavior depend on the ultraviolet, so if we change the UV modes then it modifies the IR effective theory. Changing  $a$  is changing the UV modes, so the relation between  $g_c^2$  and  $g_l^2$  will be more complex. But the relation between the theories should be analytic (or at least asymptotic) in the coupling when it is small. So the relation can be expanded;

$$g_l^2 = g_c^2 \left( 1 + c_1(a\mu)g_c^2 + c_2(a\mu)g_c^4 + \dots \right) . \quad (3)$$

To match the theories at  $O(a^0)$  we need to know the values of ALL of the coefficients  $c_i$ . To match beyond  $O(a^0)$ , say at  $O(a^2)$ , we must write down a lattice action with enough parameters to separately tune the tree level values of all  $O(a^2)$  nonrenormalizable operators consistent with cubic symmetry, and the appropriate coefficients can each be written as an expansion in  $g_c^2$  as above. In theories with fermions there are also potential  $O(a)$  nonrenormalizable operator errors.

In the early 1980's it was hoped that the series such as the one in Eq. (3) would be rapidly convergent, and perturbative computation of the first few terms would give a good match between lattice and continuum theories [21]. In fact, the lattice perturbative calculations are very complicated, and the convergence of the perturbative series is not very good. It is more practical to make nonperturbative measurements in the lattice theory and find  $g_l^2$  by matching their values to continuum (perturbative or measured) values. It is also possible to go beyond an  $O(a^0)$  matching; for instance, Lüscher's group has recently been able to match lattice Yang-Mills theory with Wilson fermions to the continuum theory, using an improved lattice action and nonperturbative matching conditions to remove all  $O(a^0)$  and  $O(a)$  errors at the nonperturbative level [22].

The idea in 3-D bosonic Yang-Mills Higgs theory is similar, but because we are in a lower dimension, the behavior of the expansion is fundamentally different. The continuum action, say, when there is a fundamental complex Higgs field, is

$$\int d^3x \left( \frac{1}{2} \text{Tr} F_{ij} F_{ij} + (D_i \phi)^\dagger D_i \phi + m_{3c}^2 \phi^\dagger \phi + \lambda_{3c} (\phi^\dagger \phi)^2 \right). \quad (4)$$

The dimensionalities of the fields and couplings are now  $[\phi] = \text{energy}^{1/2}$ ,  $[A] = \text{energy}^{1/2}$ ,  $[g^2] = [\lambda] = \text{energy}$ , and  $[m_3^2] = \text{energy}^2$ . Note that the coupling constants are dimensionful. In terms of the 4 dimensional couplings,  $g_3^2 = g_4^2 T$  at tree level.<sup>2</sup>

A lattice version of this theory has as its action

$$a^3 \sum_x \left\{ \frac{2}{g_{3l}^2 a^4} \sum_{i < j} \text{Tr} (\mathbf{1} - P_{ij}) + \frac{Z_\phi}{a^2} \sum_i \left( (\phi(x) - U_i(x) \phi(x+i))^\dagger (\phi(x) - U_i(x) \phi(x+i)) \right) + m_{3l}^2 Z_\phi \phi^\dagger(x) \phi(x) + \lambda_{3l} Z_\phi^2 (\phi^\dagger \phi)^2 \right\}. \quad (5)$$

If the theory were noninteracting then the lattice-continuum relations would be

$$g_{3l}^2 = g_{3c}^2, \quad Z_\phi = 1, \quad \lambda_{3l} = \lambda_{3c}, \quad m_{3l}^2 = m_{3c}^2, \quad (6)$$

and the measured lattice value of the operator insertion  $\langle \phi^\dagger \phi \rangle$  would match the continuum value after lattice zero point fluctuations were corrected for. (The only operator insertion we will consider in this work is  $\langle \phi^\dagger \phi \rangle$ .)

The corrections to these relations must again be analytic in  $g_3^2$ ,  $\lambda_3$ , and  $m_3^2$ . But all of these quantities are dimensionful, and their dimensionality will be balanced by powers of  $a$ . So, for instance,

$$g_{3l}^2 = g_{3c}^2 \left( 1 + c_1 g_{3c}^2 a + c_2 g_{3c}^4 a^2 + c_3 g_{3c}^2 \lambda_c a^2 + \dots \right). \quad (7)$$

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<sup>2</sup>Alternately one may maintain the 4-D normalization of the fields and couplings, but then a dimensionful factor,  $1/T$ , appears in front of the action, so  $T$  always appears along with  $g^2$  or  $\lambda$  in the loopwise expansion. This leads to the same parametric arguments we present here, as each  $g^2$  appears with a  $T$  which has to be balanced by an  $a$  for dimensional reasons.

The appearance of positive powers of  $a$  in this equation is because of superrenormalizability. Because all couplings have positive engineering dimension, the theory can be freed of errors at any particular order in  $a$  by a perturbative calculation with a finite number of diagrams.

The only corrections which can occur at  $O(a^{-1})$  are a correction to the mass,

$$\delta m_l^2 \sim g_c^2 a^{-1}, \lambda_c a^{-1} \quad (8)$$

and a constant difference between lattice and continuum values for  $\langle \phi^\dagger \phi \rangle$ ,

$$\delta \langle \phi^\dagger \phi \rangle_l \sim a^{-1}. \quad (9)$$

These are generated by one loop diagrams, and the latter is already present at zero coupling. Here and in what follows the 3 subscript will be implied, when there can be no confusion.

At  $O(a^0)$  or  $O(a^0 \ln a\mu)$ , the only corrections are of form

$$\delta m_l^2 \sim g_c^4, g_c^2 \lambda_c, \lambda_c^2, \quad (10)$$

and

$$\delta \langle \phi^\dagger \phi \rangle \sim g_c^2, \lambda_c. \quad (11)$$

The coefficients are determined by a two loop calculation and may contain  $\ln(\mu a)$ . Knowing all of these corrections allows one to make a match which is correct in the strict  $a \rightarrow 0$  limit. These counterterms are computed for SU(2) Higgs theory in [4, 5] and extended to SU( $N$ ) in [19].

At  $O(a)$ , the possible additive corrections to  $m_l^2$  and  $\langle \phi^\dagger \phi \rangle$  are

$$\delta m_l^2 \sim a g_c^6, a g_c^4 \lambda_c, a g_c^2 \lambda_c^2, a \lambda_c^3, \quad (12)$$

and

$$\delta \langle \phi^\dagger \phi \rangle \sim a g_c^4, a g_c^2 \lambda_c, a \lambda_c^2. \quad (13)$$

These arise from 3 loop diagrams. The other possible corrections are

$$m_l^2 - m_c^2 \sim a g_c^2 m_c^2, a \lambda_c m_c^2, \quad (14)$$

$$\langle \phi^\dagger \phi \rangle_l - \langle \phi^\dagger \phi \rangle_c \sim a g_c^2 \langle \phi^\dagger \phi \rangle_c, a \lambda_c \langle \phi^\dagger \phi \rangle_c, a m_c^2, \quad (15)$$

$$Z_\phi - 1 \sim a g_c^2, a \lambda_c, \quad (16)$$

$$\delta g_l^2 \sim a g_c^4, \quad (17)$$

$$\delta \lambda_l \sim a g_c^4, a g_c^2 \lambda_c, a \lambda_c^2. \quad (18)$$

These corrections are potentially generated by one loop diagrams.

We should also check whether nonrenormalizable operators are induced; the terms we might possibly need to include in the tree action are of form

$$(a g_c^2, a \lambda_c) (\phi^\dagger \phi)^3, a F^2 \phi^\dagger \phi, a \phi^\dagger \phi (D_i \phi)^2, a (\phi^\dagger \phi)^8. \quad (19)$$

But any diagram which generates one of these contains too many powers of  $g^2$  or  $\lambda$ , which must be balanced by more powers of  $a$ . Even at one loop, none of these terms are generated below  $O(a^3)$ . So the corrections listed are exhaustive at  $O(a)$ .

It would be nice to compute all  $O(a)$  corrections. But we see that two of these, additive corrections to  $m_l^2$  and  $\langle\phi^\dagger\phi\rangle$  respectively, involve 3 loop computations, which are prohibitively difficult. Hence a complete analytic treatment of all  $O(a)$  corrections appears unrealistic. Nevertheless, we accomplish something worthwhile by computing the remaining coefficients. This is because they influence and report the strength of the phase transition, while the additive corrections to  $m_l^2$  and  $\langle\phi^\dagger\phi\rangle$  do not. Computing the others eliminates  $O(a)$  errors in jumps in order parameters, the surface tension, the latent heat, the profile of the phase interface, the temperature range of metastability, etc. Only the  $\overline{\text{MS}}$  value of  $m_{3c}^2$  at  $T_c$  and the absolute  $\overline{\text{MS}}$  value of  $\langle\phi^\dagger\phi\rangle$  in one phase are determined with less accuracy. Further, it may be much easier to determine the (uncalculated) coefficients for the 3-loop  $m_l^2$  and  $\langle\phi^\dagger\phi\rangle$  corrections by Monte-Carlo methods if all other  $O(a)$  errors are eliminated, and it may be easier to compute these than it would be to eliminate all  $O(a)$  errors in all quantities by extrapolation.

Finally we should point out that the 3-D values of  $m_3^2$  and  $\langle\phi^\dagger\phi\rangle$  are known with less accuracy in terms of the 4-D parameters (and hence physical measurables) as  $g_3^2$ ,  $\lambda_3$ , and wave function normalizations. Writing

$$g_{3c}^2 = g_4^2(\mu)T \left(1 + c_1(\mu/T)g_4^2 + c_2(\mu/T)g_4^4 + \dots\right), \quad (20)$$

a 1 loop calculation of the dimensional reduction step already establishes the value of  $c_1$ , and the relative uncertainty in  $g_3^2$  is  $O(\alpha^2)$ . On the other hand, a 1 loop determination of  $m_{3c}^2/g_{3c}^4$  leaves an  $O(1)$  error, and the 2 loop determination, which is presently state of the art, leaves an  $O(\alpha)$  error.<sup>3</sup> Improving this by doing the 3 loop computation would be of comparable difficulty to the 3 loop calculation of the lattice continuum match. If we make the parametric estimate that  $g_{3c}^2 a \sim \alpha$ , ie  $a \sim 1/T$ , then the  $O(a)$  calculation of  $\delta m_{3l}^2$  is unjustified without the 3 loop dimensional reduction calculation. But the  $O(a)$  calculation of the other quantities is parametrically justified.

It is possible in principle to continue the lattice to continuum match to arbitrary order in  $a$ . At any finite order, the lattice action requires a finite number of terms and the matching requires the calculation of a finite number of diagrams. In practice the complexity rises very fast beyond  $O(a)$ . To remove all  $O(a^2)$  errors in measurables describing the strength of the phase transition, one must expand the lattice action to contain terms which can be tuned to remove all tree level  $O(a^2)$  nonrenormalizable operators. This changes the Feynman rules of the perturbation theory. One will also need to compute corrections to the wave functions and couplings at two loops, which involves over 200 topologically distinct diagrams. We will content ourselves with  $O(a)$  improvement here.

### 3 Improvement in $\text{SU}(N)$ theory

Our goal is, given a continuum  $\text{SU}(N)$  plus real adjoint Higgs plus complex fundamental Higgs theory in 3 dimensions, to write down the lattice theory which is equivalent up to  $O(a)$  additive errors in masses and expectation values of operator insertions and  $O(a^2)$  errors in

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<sup>3</sup>This is for the integration over the “superheavy” scale, ie nonzero Matsubara frequencies. If one also integrates out zero Matsubara frequency modes which take on  $O(gT)$  masses, the “heavy” scale, then  $\alpha$  in this argument becomes  $\alpha^{1/2}$ .

wave functions, couplings, multiplicative corrections to masses and operator insertions, and nonrenormalizable operators.

We denote the adjoint Higgs field as  $\Phi = \Phi^a T^a$  and the fundamental Higgs field as  $\phi$ . Our group conventions are standard, ie  $\text{Tr} T^a T^b = (1/2)\delta_{ab}$ . It is to be understood that  $D_i$  acting on  $\Phi$  is the adjoint covariant derivative and  $D_i$  acting on  $\phi$  is the fundamental covariant derivative. The Lagrangian density of the continuum theory we consider is

$$\begin{aligned} \mathcal{L}_{\text{cont}} = & \frac{1}{2} \text{Tr} F_{ij} F_{ij} + \text{Tr}(D_i \Phi D_i \Phi) + (D_i \phi)^\dagger (D_i \phi) + \\ & + m_\Phi^2 \text{Tr} \Phi^2 + m_\phi^2 \phi^\dagger \phi + \lambda_1 (\text{Tr} \Phi^2)^2 + \lambda_2 \text{Tr} \Phi^4 + \lambda (\phi^\dagger \phi)^2 + \\ & + h_1 \text{Tr} \Phi^2 \phi^\dagger \phi + \frac{h_2}{2} d_{abc} \Phi^a \Phi^b \phi^\dagger T^c \phi. \end{aligned} \quad (21)$$

Note that there are two possible adjoint field self-interactions and two possible interactions between fundamental and adjoint Higgs fields. In SU(2) and in SU(3),  $\text{Tr} \Phi^4 = (1/2)(\text{Tr} \Phi^2)^2$ . In SU(2) this is because  $d_{abc} = 0$ , and in SU(3) it is because of the relation, special to SU(3), that

$$d_{abe} d_{cde} + d_{ace} d_{bde} + d_{ade} d_{bce} = \frac{1}{3} (\delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) \text{ for SU(3)}. \quad (22)$$

Hence, in these theories the two adjoint Higgs field self-interactions are dependent, and only the combination  $(\lambda_1 + \lambda_2/2)$  is important. Also note that the  $h_2$  term can be dropped for SU(2), because  $d_{abc} = 0$  in that case.

A choice for the Lagrangian density for the lattice theory which is general enough to allow  $O(a)$  corrections is

$$\begin{aligned} \mathcal{L}_{\text{latt}}(x) = & \frac{2}{Z_g g^2 a^4} \sum_{i < j} \text{Tr}(\mathbf{1} - P_{ij}(x)) + \frac{2Z_\phi}{a^2} \sum_i [\phi^\dagger(x) \phi(x) - \phi^\dagger(x) U_i(x) \phi(x+i)] + \\ & + \frac{2Z_\Phi}{a^2} \sum_i [\text{Tr} \Phi^2(x) - \text{Tr} \Phi(x) U_i(x) \Phi(x+i) U_i^\dagger(x)] + \\ & + (\lambda + \delta\lambda) Z_\phi^2 (\phi^\dagger \phi)^2 + (\lambda_1 + \delta\lambda_1) Z_\Phi^2 (\text{Tr} \Phi^2)^2 + (\lambda_2 + \delta\lambda_2) Z_\Phi^2 \text{Tr} \Phi^4 + \\ & + (h_1 + \delta h_1) Z_\phi Z_\Phi \phi^\dagger \phi \text{Tr} \Phi^2 + \frac{(h_2 + \delta h_2)}{2} Z_\phi Z_\Phi d_{abc} \Phi^a \Phi^b \phi^\dagger T^c \phi + \\ & + \delta m_\phi^2 Z_\phi \phi^\dagger \phi + \delta m_\Phi^2 Z_\Phi \text{Tr} \Phi^2 + \\ & + \begin{bmatrix} m_\phi^2 & m_\Phi^2 \end{bmatrix} Z_m \begin{bmatrix} Z_\phi \phi^\dagger \phi & Z_\Phi \text{Tr} \Phi^2 \end{bmatrix}^T. \end{aligned} \quad (23)$$

The couplings and masses here are the continuum theory values, and the coefficients  $Z_g$ ,  $Z_\Phi$ ,  $Z_\phi$ ,  $\delta\lambda_i$ ,  $\delta m_i^2$ , and  $Z_m$  perform the  $O(a)$  corrections.  $Z_m$  is a  $2 \times 2$  matrix, which differs at  $O(a)$  from the identity matrix.

Further, to convert measured values of operator insertions to the equivalent continuum values, we need an additive and a multiplicative renormalization of the operator insertions,

$$\begin{bmatrix} \langle \phi^\dagger \phi \rangle \\ \langle \text{Tr} \Phi^2 \rangle \end{bmatrix}_{\text{continuum}} = \begin{bmatrix} \delta \langle \phi^\dagger \phi \rangle \\ \delta \langle \text{Tr} \Phi^2 \rangle \end{bmatrix} + Z_{\text{OP}} \begin{bmatrix} \langle \phi^\dagger \phi \rangle \\ \langle \text{Tr} \Phi^2 \rangle \end{bmatrix}_{\text{latt, measured}}. \quad (24)$$

Here  $Z_{\text{OP}}$  is also a matrix, with  $O(a)$  off diagonal elements which account for mixing between the operator insertions. All notations are chosen to follow [19]; to convert to the notation

of [13], change the signs on the counterterms in the scalar effective potential, introduce  $\lambda_{L,i} = 4\lambda_i/g^2$ , rescale the scalar fields by a factor of  $g^2 a T/4$  and the  $m^2$  by a factor of  $a^2$ , and write  $4/g^2 a \equiv \beta_L$ . Also,  $Z_g$  was called  $Z_A^{-1}$  there.

Neither  $Z_\phi$ ,  $Z_\Phi$ , nor  $Z_{\text{OP}}$  are separately gauge invariant or physical; one may for instance rescale the  $\Phi$  field by a factor of  $\eta = 1 + O(a)$  and change  $Z_\Phi$  by  $\eta^{-2}$  and the right column of  $Z_{\text{OP}}$  by  $\eta^{-2}$  without changing anything. It is the matrix  $Z_{\text{OP}} \times \text{Diag}[Z_\phi^{-1} \quad Z_\Phi^{-1}]$  which is gauge invariant. This combination equals  $Z_m$ , a statement which will remain true at all orders.

Similarly, while  $Z_g$  is gauge invariant, if one tries to relate the 3-D theory gauge field  $A$  to the lattice gauge field  $A$  defined by  $U_i(x) = \exp(-iagT^a A_i^a(x + i/2))$ , one will find the relation contains a non-gauge invariant  $O(a)$  correction, for the same reason that the 4-D continuum gauge field wave function renormalization is not gauge invariant, while the renormalization of the coupling constant is. This is only important if one is interested in measuring  $A$  field correlators in some gauge, and we will not compute the  $A$  field wave function correction here.

To determine all of the required corrections, one must calculate all 1 loop self-energy insertions for all fields to  $O(p^2)$ , all 1 loop  $O(p)$  corrections to one scalar-gauge vertex, all 1 loop scalar 4-point corrections at zero external momentum, all 1 loop corrections to the insertion of a  $\phi^\dagger\phi$  or  $\text{Tr}\Phi^2$  operator on a zero momentum scalar line, all 1 and 2 loop scalar self-energy insertions at zero momentum, and all 1 and 2 loop vacuum diagrams with  $\phi^\dagger\phi$  or  $\text{Tr}\Phi^2$  insertions. The topologically distinct diagrams required in Landau gauge are presented in Figure 1. In a general gauge, additional diagrams are needed. The Feynman rules are spread between [23, 5, 13].

All of the diagrams have been computed previously: (a) at  $O(p^0)$  in [24], (b) in [4], (c) in [5],<sup>4</sup> and (a) at  $O(p^2)$ , (d) in [13]. Our task here is purely combinatoric. We must include 2 types of scalars and compute the appropriate  $\text{SU}(N)$  group factors. All the group theoretic identities which are needed appear in Appendix A of [18]. The converse is also true.

Rather than present the full combinatorial details of the calculation, we will skip to the results. Two numerical constants appear, and their values are  $\Sigma = 3.175911536$ ,  $\xi = 0.152859325$ . Also, when comparing the dimensionality of the right and lefthand sides, remember that the coupling constants  $g^2$ ,  $\lambda_i$  are in 3-D notation and correspond to  $T$  times the dimensionless 4-D values (at some renormalization point). The coupling renormalizations are<sup>5</sup>

$$Z_g^{-1} - 1 = \frac{g^2 a}{4\pi} \left( \frac{2\pi}{9N} (2N^2 - 3) + N \left( \frac{37\xi}{12} - \frac{\pi}{9} \right) + N \left[ \frac{\Sigma}{24} - \frac{\xi}{6} \right] + \left\{ \frac{\Sigma}{24} - \frac{\xi}{6} \right\} \right), \quad (25)$$

$$\begin{aligned} \delta\lambda = \frac{a}{4\pi} & \left( \frac{N^3 + N^2 - 4N + 2}{4N^2} g^4 \xi + (2N + 8) \lambda^2 \xi + \frac{N^2 - 1}{4} h_1^2 \xi + \right. \\ & \left. + \frac{(N^2 - 4)(N - 1)}{8N^2} h_2^2 \xi - \frac{N^2 - 1}{6N} (18\xi + \Sigma) g^2 \lambda \right), \end{aligned} \quad (26)$$

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<sup>4</sup>The computation there is not organized in terms of two loop scalar self-energy insertions, but it is equivalent.

<sup>5</sup>We choose to report  $Z_g^{-1} - 1$  because at two loops this quantity is only changed by two loop self-energy diagrams and vertex corrections, ie it absorbs iterated insertions of 1 loop self-energy diagrams on propagators. But at  $O(a)$  accuracy we could equally have written Eq. (25) as a correction to  $1 - Z_g$ .



$$\delta\lambda_1 = \frac{a}{4\pi} \left( 3g^4\xi + (N^2 + 7)\lambda_1^2\xi + \frac{4N^2 - 6}{N}\lambda_1\lambda_2\xi + \frac{3N^2 + 9}{N^2}\lambda_2^2\xi + \frac{N}{2}h_1^2\xi - \frac{1}{2N}h_2^2\xi - \frac{N}{3}(18\xi + \Sigma)g^2\lambda_1 \right), \quad (27)$$

$$\delta\lambda_2 = \frac{a}{4\pi} \left( Ng^4\xi + 2\frac{N^2 - 9}{N}\lambda_2^2\xi + 12\lambda_1\lambda_2\xi + \frac{1}{2}h_2^2\xi - \frac{N}{3}(18\xi + \Sigma)g^2\lambda_2 \right), \quad (28)$$

$$\delta h_1 = \frac{a}{4\pi} \left( 2g^4\xi + (N^2 + 1)h_1\lambda_1\xi + (2N + 2)h_1\lambda\xi + 2h_1^2\xi + \frac{N^2 - 4}{N}h_2^2\xi + \frac{2N^2 - 3}{N}\lambda_2h_1\xi - \frac{3N^2 - 1}{12N}(18\xi + \Sigma)g^2h_1 \right), \quad (29)$$

$$\delta h_2 = \frac{a}{4\pi} \left( Ng^4\xi + h_2 \left( 2\lambda + 2\lambda_1 + \frac{N^2 - 6}{N}\lambda_2 + 4h_1 + \frac{N^2 - 12}{2N}h_2 \right) \xi - \frac{3N^2 - 1}{12N}(18\xi + \Sigma)g^2h_2 \right). \quad (30)$$

For the theory without the adjoint Higgs field, drop the term with square brackets in Eq. (25), set  $\lambda_1 = \lambda_2 = h_1 = h_2 = 0$  in the equation for  $\delta\lambda$ , and ignore the equations for  $\delta\lambda_1$  etc. Putting  $N = 2$ , we recover the results of [13], though the notation there is somewhat different. For the theory without the fundamental Higgs, drop the term in curly brackets in Eq. (25), set  $\lambda = h_1 = h_2 = 0$ , and ignore the equations for their corrections.

The fundamental scalar contribution to Eq. (25) is wrong in [13] by a factor of  $1/2$ , both for  $SU(2)$  and  $U(1)$ . This error is not numerically important; even in  $SU(2)$  it leads to a 3% error in  $Z_g^{-1} - 1$ . Appendix C of that paper, which treats the case with both Higgs fields for  $N = 2$ , has several typographical omissions of factors of  $\Sigma/4\pi$  and  $\xi/4\pi$ , and also the  $g^4$  contributions to  $(\delta\lambda_1 + \delta\lambda_2/2)$  and to  $\delta h_1$  are both off by a factor of 2 there, due to a combinatorial error. These are the only combinatorial mistakes in that paper.<sup>6</sup>

Before continuing, we mention two checks on the calculation so far. First, both for  $N = 2$  and  $N = 3$ , the combination  $(\lambda_1 + \lambda_2/2)$  is important, not  $\lambda_1$  and  $\lambda_2$  separately. Checking, one finds that  $(\delta\lambda_1 + \delta\lambda_2/2)$  is fixed when one varies  $\lambda_1$  keeping  $(\lambda_1 + \lambda_2/2)$  fixed, both for  $N = 2$  or  $N = 3$ . Also,  $h_2$  should have no influence when  $N = 2$ . Indeed, its contribution to the rescaling of other couplings vanishes for  $N = 2$ .

Now let us present  $Z_\Phi$ ,  $Z_\phi$ , and  $Z_{OP}$ . Since they are not separately gauge invariant, we will present them in Landau gauge. In this gauge, at one loop, the corrections to wave functions all go as  $g^2$  and the corrections to  $Z_{OP}$  depend on scalar self-couplings. This statement does not persist at 2 loops. The results are

$$Z_\phi - 1 = \frac{g^2 a}{4\pi} \frac{N^2 - 1}{12N} (18\xi + \Sigma), \quad (31)$$

$$Z_\Phi - 1 = \frac{g^2 a}{4\pi} \frac{N}{6} (18\xi + \Sigma), \quad (32)$$

$$Z_{OP} - \mathbf{1} = \frac{\xi a}{4\pi} \left[ \begin{array}{cc} (2N + 2)\lambda & Nh_1 \\ \frac{N^2 - 1}{2}h_1 & (N^2 + 1)\lambda_1 + \frac{2N^2 - 3}{N}\lambda_2 \end{array} \right]. \quad (33)$$

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<sup>6</sup>However, early preprint versions had further mistakes which were corrected before publication.

Here  $\mathbf{1}$  is the  $2 \times 2$  identity matrix. Varying the gauge changes the wave function corrections and the on diagonal elements of  $Z_{OP}$ , but not  $Z_m$ . The off diagonal elements of  $Z_{OP}$  are also gauge invariant at  $O(a)$ , but this will not persist at higher loop orders.

It remains to present  $\delta m_\phi^2$ ,  $\delta m_\Phi^2$ ,  $\delta \langle \phi^\dagger \phi \rangle$ , and  $\delta \langle \text{Tr} \Phi^2 \rangle$ . The one loop corrections are [19]

$$\delta m_\phi^2(\text{one loop}) = -\frac{\Sigma}{4\pi a} \left( \frac{N^2-1}{N} g^2 + 2(N+1)\lambda + \frac{N^2-1}{2} h_1 \right), \quad (34)$$

$$\delta m_\Phi^2(\text{one loop}) = -\frac{\Sigma}{4\pi a} \left( 2Ng^2 + Nh_1 + (N^2+1)\lambda_1 + \frac{2N^2-3}{N}\lambda_2 \right), \quad (35)$$

$$\delta \langle \phi^\dagger \phi \rangle(\text{one loop}) = -N \left( \frac{\Sigma}{4\pi a} + \frac{\xi m_\phi^2 a}{4\pi} \right), \quad (36)$$

$$\delta \langle \text{Tr} \Phi^2 \rangle(\text{one loop}) = -\frac{N^2-1}{2} \left( \frac{\Sigma}{4\pi a} + \frac{\xi m_\Phi^2 a}{4\pi} \right). \quad (37)$$

We have included the  $O(a)$ ,  $m^2$  dependent correction to the operator insertion expectation values. For the case where there is one Higgs field but not the other, set  $h_1 = 0$ . Also note that our  $h_1$  differs by a factor of 2 in normalization from the parameter  $h_3$  appearing in [5].

The two loop corrections have been computed in [19], for the case where there is an adjoint scalar, or a fundamental scalar, but not both. For our case, we must add new terms when there are two types of scalars, and contributions from inserting  $O(a)$  counterterms into 1 loop diagrams.

The two loop contributions listed in [19] are

$$\begin{aligned} \delta m_\phi^2 + = & -\frac{N^2-1}{16\pi^2} \left\{ \left[ g^4 \frac{4N^2-N+3}{4N^2} + 2\lambda g^2 \frac{N+1}{N} - 4\lambda^2 \frac{1}{N-1} \right] \left( \ln \frac{6}{a\mu} + \zeta \right) + \right. \\ & + 2\lambda g^2 \frac{N+1}{N} \left( \frac{\Sigma^2}{4} - \delta \right) + g^4 \frac{1}{4N^2} \left[ \frac{4N^2-1}{4} \Sigma^2 + \frac{3N^2-8}{3} \pi \Sigma + N^2 + \right. \\ & \left. \left. + 1 - 4N(N+1)\rho - 2(3N^2-1)\delta + 2N^2(2\kappa_1 - \kappa_4) \right] \right\}, \end{aligned} \quad (38)$$

$$\begin{aligned} \delta m_\Phi^2 + = & -\frac{1}{16\pi^2} \left\{ \left[ 2Ng^2 \left( (N^2+1)\lambda_1 + \frac{2N^2-3}{N}\lambda_2 \right) - 2(N^2+1)\lambda_1^2 - \right. \right. \\ & - \frac{4(2N^2-3)}{N}\lambda_1\lambda_2 - \frac{N^4-6N^2+18}{N^2}\lambda_2^2 \left. \right] \left( \ln \frac{6}{a\mu} + \zeta \right) + \\ & + 2Ng^2 \left( \frac{\Sigma^2}{4} - \delta \right) \times \left( (N^2+1)\lambda_1 + \frac{2N^2-3}{N}\lambda_2 \right) + \\ & \left. + g^4 N^2 \left[ \frac{5\Sigma^2}{8} + \frac{3N^2-8}{6N^2} \pi \Sigma - 4(\delta + \rho) + 2\kappa_1 - \kappa_4 \right] \right\}, \end{aligned} \quad (39)$$

$$\delta \langle \phi^\dagger \phi \rangle + = -\frac{g^2}{16\pi^2} (N^2-1) \left( \ln \frac{6}{a\mu} + \zeta + \frac{\Sigma^2}{4} - \delta \right), \quad (40)$$

$$\delta \langle \text{Tr} \Phi^2 \rangle + = -\frac{g^2}{16\pi^2} N(N^2-1) \left( \ln \frac{6}{a\mu} + \zeta + \frac{\Sigma^2}{4} - \delta \right), \quad (41)$$

where the newly introduced constants are  $\zeta = .08849$ ,  $\delta = 1.942130$ ,  $\rho = -.313964$ ,  $\kappa_1 = .958382$ ,  $\kappa_4 = 1.204295$ . Here and in the following we will write  $+ =$  to show that these contributions are to be added to those listed previously.

In the case where both Higgs fields are present, there are added contributions to the mass counterterms of

$$\delta m_\phi^2 \quad + = \quad -\frac{N^2 - 1}{16\pi^2} \left\{ \left( -\frac{1}{2}h_1^2 - \frac{N^2 - 4}{4N^2}h_2^2 + Nh_1g^2 - \frac{1}{4}g^4 \right) \times \left( \ln \frac{6}{a\mu} + \zeta \right) + \left( \frac{\Sigma^2}{4} - \delta \right) Ng^2h_1 - \rho g^4 \right\}, \quad (42)$$

$$\delta m_\Phi^2 \quad + = \quad -\frac{1}{16\pi^2} \left\{ \left( -Nh_1^2 - \frac{N^2 - 4}{2N}h_2^2 + (N^2 - 1)h_1g^2 - \frac{N}{2}g^4 \right) \times \left( \ln \frac{6}{a\mu} + \zeta \right) + \left( \frac{\Sigma^2}{4} - \delta \right) (N^2 - 1)h_1g^2 - 2N\rho g^4 \right\}. \quad (43)$$

At  $N = 2$  these match the results of [5]. Do not include these in the theory with only a fundamental or an adjoint Higgs, but not both.

In addition, there are contributions at  $O(a^0)$  arising from  $O(a)$  counterterm insertions into diagrams which give  $O(1/a)$  divergences. These lead to the following corrections:

$$\delta m_\phi^2 \quad + = \quad -\frac{\Sigma}{4\pi a} \left( \frac{N^2 - 1}{N}g^2(Z_g - 1) + 2(N + 1)\delta\lambda + \frac{N^2 - 1}{2}\delta h_1 \right), \quad (44)$$

$$\delta m_\Phi^2 \quad + = \quad -\frac{\Sigma}{4\pi a} \left( 2Ng^2(Z_g - 1) + N\delta h_1 + (N^2 + 1)\delta\lambda_1 + \frac{2N^2 - 3}{N}\delta\lambda_2 \right), \quad (45)$$

$$\begin{bmatrix} \delta\langle\phi^\dagger\phi\rangle \\ \delta\langle\text{Tr}\Phi^2\rangle \end{bmatrix} \quad + = \quad (Z_m - \mathbf{1}) \begin{bmatrix} \delta\langle\phi^\dagger\phi\rangle \\ \delta\langle\text{Tr}\Phi^2\rangle \end{bmatrix} (1 \text{ loop}). \quad (46)$$

The  $O(a)$  counterterms appearing here were given earlier, and as usual the term in the parenthesis in Eq. (46) is to be interpreted as a  $2 \times 2$  matrix. It should be clear how to pare these equations down to the case where there is only one type of Higgs field.

This completes the calculation of all counterterms needed for the  $O(a)$  improvement of the theory, except for the additive  $O(a)$  corrections to the masses and operator insertions which arise at 3 loops, which we will not calculate, as advertized.

What should you do, though, if you have already taken data without including these corrections in the Lagrangian? That is, what if you have data for a theory which naively has a lattice spacing  $a_n$ , no wave function or  $g$  renormalization, and naive couplings  $\lambda_n$ , etc? Then you should figure out what theory you were “really” looking at, and reinterpret the results accordingly. By demanding that the “naive” action

$$\begin{aligned} a_n^3 \sum_x \quad & \left\{ \frac{2}{g^2 a_n^4} \sum_{i < j} \text{Tr}(\mathbf{1} - P_{ij}(x)) + \frac{2}{a_n^2} \sum_i [\phi_n^\dagger(x)\phi_n(x) - \phi_n^\dagger(x)U_i(x)\phi_n(x+i)] + \right. \\ & + \frac{2}{a_n^2} \sum_i [\text{Tr}\Phi_n^2(x) - \text{Tr}\Phi_n(x)U_i(x)\Phi_n(x+i)U_i^\dagger(x)] + \\ & \left. + \lambda_n(\phi_n^\dagger\phi_n)^2 + \lambda_{1n}(\text{Tr}\Phi_n^2)^2 + \lambda_{2n}\text{Tr}\Phi_n^4 + \right. \end{aligned}$$

$$\begin{aligned}
& +h_{1n}\phi_n^\dagger\phi_n\text{Tr}\Phi_n^2 + \frac{h_{2n}}{2}d_{abc}\Phi_n^a\Phi_n^b\phi_n^\dagger T^c\phi_n + \\
& +(m_{\phi,n}^2 + \delta m_{\phi,n}^2)\phi_n^\dagger\phi_n + \\
& +(m_{\Phi,n}^2 + \delta m_{\Phi,n}^2)\text{Tr}\Phi_n^2 \Big\}
\end{aligned} \tag{47}$$

be equal to  $a^3 \sum_x \mathcal{L}_{\text{latt}}(x)$  of Eq. (23), one finds that the naive values labeled with the  $n$  subscript are related to the  $O(a)$  corrected values by

$$a = Z_g^{-1}a_n, \tag{48}$$

$$\phi = (Z_g Z_\phi^{-1})^{1/2}\phi_n \tag{49}$$

$$\Phi = (Z_g Z_\Phi^{-1})^{1/2}\Phi_n \tag{50}$$

$$\lambda = Z_g\lambda_n - \delta\lambda, \tag{51}$$

$$\lambda_1 = Z_g\lambda_{1n} - \delta\lambda_1, \tag{52}$$

$$\lambda_2 = Z_g\lambda_{2n} - \delta\lambda_2, \tag{53}$$

$$h_1 = Z_g h_{1n} - \delta h_1, \tag{54}$$

$$h_2 = Z_g h_{2n} - \delta h_2, \tag{55}$$

$$\delta m_\phi^2 = Z_g^2 \delta m_{\phi,n}^2, \tag{56}$$

$$\delta m_\Phi^2 = Z_g^2 \delta m_{\Phi,n}^2, \tag{57}$$

$$\begin{bmatrix} m_\phi^2 \\ m_\Phi^2 \end{bmatrix} = Z_g^2 Z_m^{-1} \begin{bmatrix} m_{\phi,n}^2 \\ m_{\Phi,n}^2 \end{bmatrix}. \tag{58}$$

Note that, up to  $O(a)$  corrections,  $\delta m_\phi^2$  computed using  $a_n$  and the naive scalar self-couplings, and without including Eq. (44), differs from  $\delta m_\phi^2$  computed using the improved values and including Eq. (44), by precisely a factor of  $Z_g^{-2}$ , just what is required above.<sup>7</sup> This just says that the mass renormalization computed in [5, 19] is the right one if you use the “naive” couplings and wave functions.

Next we should relate the  $O(a)$  corrected value of the continuum operator insertion expectation values, through the  $O(a)$  corrected lattice values, to the uncorrected lattice values. We have

$$\begin{aligned}
\begin{bmatrix} \langle \phi^\dagger \phi \rangle \\ \langle \text{Tr} \Phi^2 \rangle \end{bmatrix}_{\text{continuum}} &= \begin{bmatrix} \delta \langle \phi^\dagger \phi \rangle \\ \delta \langle \text{Tr} \Phi^2 \rangle \end{bmatrix} + Z_{\text{OP}} \begin{bmatrix} \langle \phi^\dagger \phi \rangle \\ \langle \text{Tr} \Phi^2 \rangle \end{bmatrix}_{\text{latt, measured}} \\
&= Z_g Z_m \begin{bmatrix} \langle \phi_n^\dagger \phi_n \rangle_{\text{meas}} + (\delta \langle \phi^\dagger \phi \rangle)_n \\ \langle \text{Tr} \Phi_n^2 \rangle_{\text{meas}} + (\delta \langle \text{Tr} \Phi^2 \rangle)_n \end{bmatrix}.
\end{aligned} \tag{59}$$

Again, using  $a_n$  in Eqs. (36,37) and skipping Eq. (46) leads to a value for the naive counterterm which matches the corrected counterterm in the above. So if one has “naive” measured values for condensates, with the additive counterterms from [4, 5, 19] already taken out, then one applies the  $O(a)$  corrections by multiplying by  $Z_g Z_m$ , where  $Z_m$  is to be understood as a  $2 \times 2$  matrix, as usual.

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<sup>7</sup>To see this, note that one factor of  $Z_g$  comes from using  $a$  rather than  $a_n$  in Eqs. (34,35) and the other comes because Eqs. (44,45) multiply the gauge part by  $Z_g$ , while the scalar part goes as  $\lambda + \delta\lambda = Z_g\lambda_n$ .

So to apply  $O(a)$  corrections to data taken using a “naive” lattice Lagrangian, interpret the true lattice spacing as  $Z_g^{-1}a_n$ , the true scalar self-couplings as  $Z_g\lambda_n - \delta\lambda$ , the true mass as  $m^2 = Z_m^{-1}Z_g^2m_n^2$ , and the true value of order parameters as  $Z_mZ_g$  times the naive values. Note that  $Z_g < 1$  and  $(Z_g - 1)\lambda - \delta\lambda < 0$ , so jumps in order parameters are generally smaller than the “naive” values and the true scalar self-couplings are generally smaller than the “naive” values. Using the “naive” values tends to over-report the strength of the phase transition. Also remember that the mass correction should only be used on the difference in 3-D mass parameter between two trials with the same lattice spacing and scalar couplings, and the order parameter correction should only be used on the difference in the order parameter between phases or between trials with different 3-D masses (and here one should also be careful to include the mass dependent term in Eqs. (36,37)). This is because we have not computed the  $O(a)$  additive corrections to the mass and order parameter counterterms.

To correct the latent heat and the surface tension, note that they are energies per unit volume and area, respectively; the  $O(a)$  corrections come about from the relation between  $a$  and  $a_n$ . Similarly, correlation lengths are corrected by converting from lattice units to physical units using  $a$  rather than  $a_n$ . If one is interested in operator insertions not discussed here, it will be necessary to extend this work by computing the renormalizations of those insertions.

We will present numerical values of renormalization constants in some particular theories, and give step by step instructions for building  $O(a)$  improved Lagrangians, in Appendix B.

One final comment is in order. As we have written them, the corrections  $Z_m$ ,  $Z_g$ ,  $\delta\lambda$  etc which are needed for this  $O(a)$  improvement are to be computed with the  $O(a)$  improved parameters  $\lambda$ ,  $a$ , etc, not with  $\lambda_n$ ,  $a_n$ , etc. If one only knows the “naive” values, this might require an iterative or “bootstrap” type calculation. However, the difference between using the improved and unimproved parameters in the calculation is formally  $O(a^2)$ , so one should not worry too much which is used—except for computing the one loop  $O(1/a)$  corrections to masses and order parameters, as already discussed.

## 4 What to do in $SU(N) \times SU(M)$

Sometimes it is interesting to treat a case where the gauge group is not simple. Here we will restrict our attention to  $SU(N) \times SU(M)$ , where each scalar field is a singlet under one or the other group, and we intend to be illustrative but not exhaustive, so we will consider the case where an  $SU(N)$  group has both an adjoint and a fundamental scalar (it is easy to delete one of them) and there is one more scalar which transforms trivially under  $SU(N)$  but perhaps nontrivially under the other group, and has self-interactions. Calling the added scalar  $s$ , and writing  $s^2$  to mean  $2s^\dagger s$  if it is fundamental and  $2\text{Tr}s^2$  if it is adjoint, the new interactions between the  $s$  field and the scalar fields  $\phi$  and  $\Phi$  which can appear in the Lagrangian are

$$\frac{h_{s1}}{2}s^2\phi^\dagger\phi + \frac{h_{s2}}{2}s^2\text{Tr}\Phi^2. \quad (60)$$

If  $h_{s1} = 0 = h_{s2}$  then the theory decouples into two noninteracting parts, the  $SU(N)$  part and the part which contains the scalar  $s$ . We assume that one can compute the corrections in this limit by using the last section, and we will compute the new corrections which must

be added at nonzero  $h_{s1}$  or  $h_{s2}$ . We will compute the corrections to the couplings of the  $SU(N)$  sector and  $h_{s1}$ ,  $h_{s2}$  with the minimum information about the scalar  $s$ . To compute corrections to the sector containing the  $s$  field, just think of it as the  $SU(N)$  sector and use these same results.

We need only a little information about the scalar  $s$  and its (gauge and self) couplings to compute the corrections to the  $SU(N)$  sector. What we need to know is the number of degrees of freedom  $s$  contains,  $N_{\text{DOF}}$  (in  $SU(M)$  this is  $2M$  for a complex fundamental field and  $M^2 - 1$  for a real adjoint field), and its one loop mass renormalization at  $h_{s1} = h_{s2} = 0$ ,

$$\delta m_s^2(\text{one loop}, \text{zero } h_{s1}, h_{s2}) = -\frac{\Sigma}{4\pi a} (C_1 + C_2), \quad (61)$$

where  $C_1$  is the contribution from all scalar couplings and  $C_2$  is the contribution from any gauge couplings it participates in. They have the same units as a coupling constant.

The one loop mass corrections due to the new particle are

$$\delta m_\phi^2 \quad + = \quad -\frac{\Sigma}{4\pi a} \frac{N_{\text{DOF}}}{2} h_{s1}, \quad (62)$$

$$\delta m_\Phi^2 \quad + = \quad -\frac{\Sigma}{4\pi a} \frac{N_{\text{DOF}}}{2} h_{s2}. \quad (63)$$

there are no new one loop additive operator insertion corrections.

The one loop  $O(a)$  corrections to scalar self-couplings are increased by

$$\delta\lambda \quad + = \quad \frac{\xi a}{4\pi} \frac{N_{\text{DOF}}}{4} h_{s1}^2, \quad (64)$$

$$\delta\lambda_1 \quad + = \quad \frac{\xi a}{4\pi} \frac{N_{\text{DOF}}}{4} h_{s2}^2, \quad (65)$$

$$\delta\lambda_2 \quad + = \quad 0, \quad (66)$$

$$\delta h_1 \quad + = \quad \frac{\xi a}{4\pi} \frac{N_{\text{DOF}}}{2} h_{s1} h_{s2}, \quad (67)$$

$$\delta h_2 \quad + = \quad 0, \quad (68)$$

$$\begin{aligned} \delta h_{s1} \quad = \quad & \frac{a}{4\pi} \left\{ \left( C_1 h_{s1} + 2(N+1)\lambda h_{s1} + \frac{N^2-1}{2} h_1 h_{s2} + 2h_{s1}^2 \right) \xi - \right. \\ & \left. - \left( \frac{N^2-1}{12N} g^2 + \frac{C_2}{12} \right) (18\xi + \Sigma) h_{s1} \right\}, \end{aligned} \quad (69)$$

$$\begin{aligned} \delta h_{s2} \quad = \quad & \frac{a}{4\pi} \left\{ \left( C_1 h_{s2} + (N^2+1)\lambda_1 h_{s2} + \frac{2N^2-3}{N} \lambda_2 h_{s2} + N h_1 h_{s1} + 2h_{s2}^2 \right) \xi - \right. \\ & \left. - \left( \frac{N}{6} g^2 + \frac{C_2}{12} \right) (18\xi + \Sigma) h_{s2} \right\}. \end{aligned} \quad (70)$$

There are no corrections to the gauge field renormalization  $Z_g$  at this order. Such corrections actually start at  $O(a^3)$ .

Moreover, we should now think of  $Z_{OP}$  as a  $3 \times 3$  matrix, which at  $h_{s1} = 0 = h_{s2}$  is nonzero in the upper  $2 \times 2$  block, where it equals  $Z_{OP}$  of the last chapter, and the lower

right component, where it is  $Z_{OP}$  for  $s$ . There are no new contributions in these blocks. The terms we should compute are

$$Z_{OP} - \mathbf{1} = \frac{\xi}{4\pi a} \begin{bmatrix} \text{same} & \text{same} & Nh_{s1} \\ \text{same} & \text{same} & \frac{N^2-1}{2}h_{s2} \\ \frac{N_{\text{DOF}}}{2}h_{s1} & \frac{N_{\text{DOF}}}{2}h_{s2} & \text{same} \end{bmatrix}, \quad (71)$$

where “same” means the same value as at  $h_{s1} = 0 = h_{s2}$ . At one loop there are no corrections to wave function renormalizations due to  $s$ . These will first appear at  $O(a^2)$ .

What remains is to compute two loop mass and operator insertion additive corrections. The only new correction to the operator insertion counterterm is that the new value of  $Z_{OP}$ , Eq. (71), should be used in Eq. (46). The new corrections to the mass counterterm, aside from using the new values for  $\delta\lambda$  etc. in Eqs. (44,45), are

$$\delta m_\phi^2 \quad + = \quad -\frac{N_{\text{DOF}}}{2} \left[ \frac{\Sigma}{4\pi a} \delta h_{s1} + \frac{h_{s1}}{16\pi^2} \left( (C_2 - h_{s1}) \left( \ln \frac{6}{a\mu} + \zeta \right) + C_2 \left( \frac{\Sigma^2}{4} - \delta \right) \right) \right], \quad (72)$$

$$\delta m_\Phi^2 \quad + = \quad -\frac{N_{\text{DOF}}}{2} \left[ \frac{\Sigma}{4\pi a} \delta h_{s2} + \frac{h_{s2}}{16\pi^2} \left( (C_2 - h_{s2}) \left( \ln \frac{6}{a\mu} + \zeta \right) + C_2 \left( \frac{\Sigma^2}{4} - \delta \right) \right) \right]. \quad (73)$$

If there are more than one scalar which are  $SU(N)$  singlets, the corrections are just the sum of the corrections from each particle, except for cross terms in the renormalization of the  $h_s$  type interactions, which can be figured out from Eqs. (69,70). Things become more complicated at higher loop order, however. If there are 3 or more gauge groups, each with scalars, then the combinatorics of the one loop scalar coupling corrections get messy and what we show here may not be sufficiently general. But we know of no physically interesting theories in this class. Also, if there are scalars which transform nontrivially under more than one gauge group, the situation becomes considerably more complicated. For instance, the one loop self-coupling and two loop mass counterterms then contain  $g_1^2 g_2^2$  type terms. The particularly simple case of  $SU(2) \times U(1)$  is treated in [13, 19]. We will not attempt anything more general here.

## 5 Conclusion

We have discussed why it is possible to write  $O(a)$  corrected actions for 3 dimensional lattice Yang-Mills Higgs theories, and we have extended the previous results for  $SU(2) \times U(1)$  to  $SU(N)$  with a fundamental and/or an adjoint Higgs field. We have also shown how the corrections can often be applied to uncorrected data “after the fact.” Generally the result is that the actual value of 3-D scalar self-coupling used is lower than the naive coefficient put into the lattice calculation, and actual jumps in the order parameters are smaller than the uncorrected data imply.

The limitation of the work here is that we have not been able to compute the  $O(a)$  additive corrections to the Higgs field masses or the  $\phi^2$  operator insertions. Computing these is possible in principle but requires doing a lot of very nasty 3 loop diagrams. We have argued that for many purposes these are the least important  $O(a)$  errors, and that the knowledge of the relation between the 3 dimensional and physical, thermal 4 dimensional

theories is in any case weaker here. The corrections for the other coefficients are definitely valuable for numerical efforts to study hot gauge theories nonperturbatively, allowing better precision with less numerical effort.

A final issue we should address is: how small should  $a$  be made so that we don't have to worry about the remaining  $O(a^2)$  errors? Of course, the answer depends on what one is doing. For instance, if one is studying the strength of a phase transition where it is strong, say at very small  $\lambda$  or  $\lambda_1$ , then the natural scale of physics involved becomes shorter than  $1/g^2$  and nonrenormalizable operators may be a problem. One can estimate the importance of these operators by including them in a perturbative calculation of the effective potential, as was done in [13]. The conclusion is that nonrenormalizable derivative terms are less important than the  $O(a)$  corrections  $\delta\lambda$  to the scalar self-couplings, roughly by a factor of  $ga\sqrt{\phi^\dagger\phi}$ . If neglecting the  $O(a)$  scalar self-coupling corrections would make a significant difference and  $ga\sqrt{\phi^\dagger\phi}$  is not very small, then  $a$  is not small enough.

Away from the case where the phase transition is strong, the two loop,  $O(a^2)$  corrections to the Lagrangian parameters may also be important. We expect that the largest of these will be “tadpole” type corrections to the gauge field normalization, higher loop analogs of the first term in Eq. (25). This term arises from a nonrenormalizable operator in the action of form  $\text{Tr}F_{ij}^4$ . It turns out to dominate  $Z_g^{-1} - 1$ , which in turn is the largest of the  $O(a)$  corrections. We might expect that “tadpole” type corrections, from multiple appearances of this term and from the term of form  $\text{Tr}F_{ij}^6$ , give the most important contributions to  $Z_g^{-1} - 1$  at two loops, and that this is again the largest  $O(a^2)$  correction. (This would follow a fairly general pattern of behavior for lattice gauge theories.) Fortunately, it is not too hard to compute this limited class of 2 loop diagrams; we do so in Appendix A. The conclusion is that the two loop correction to  $Z_g^{-1} - 1$  is about the same size as the square of the one loop correction. So if the one loop correction is 10%, then the two loop correction is 1%. If nonrenormalizable operators are not a problem, then the one loop improvement is likely to be sufficient as long as its coefficients make corrections in the 10% range.

It may not always be possible to reduce  $O(a^2)$  errors to the level desired, and sometimes it is also important to know mass parameters with high precision. In these cases it may still be necessary to perform an extrapolation over data at several values of  $a$ . We expect that the  $O(a)$  corrections will still be useful in this case, because they should make the extrapolation much better behaved.

#### Acknowledgements

I am indebted to Keijo Kajantie for suggesting this topic and for correspondence, and to Mikko Laine, Arttu Rajantie, Kari Rummukainen, and particularly Misha Shaposhnikov for correspondence or discussions.

## A Two loop tadpole type diagrams

In this appendix we compute two loop “tadpole” corrections to the gauge field self-energy. By “tadpole” corrections we mean corrections emerging from a certain restricted subset of the interaction terms generated by the gauge field term in the action. Namely, we approximate that  $P_{ij} = \exp(iga^2 T^a F_{ij}^a)$ , where  $F_{ij}$  is linear in the gauge field, ie in Fourier space it is



$F_{ij}^a(k) = k_i A_j^a(k) - k_j A_i^a(k)$ . This amounts to neglecting appearances of  $f_{abc}$  in favor of  $\delta_{ab}$  and  $d_{abc}$ .

The terms in the action which emerge from this approximation are

$$a^3 \sum_x \left( \sum_{i \neq j} \frac{1}{2Z_g} \text{Tr} F_{ij}^2 - \sum_{i \neq j} \frac{g^2 a^4}{24Z_g} \text{Tr} F_{ij}^4 + \sum_{i \neq j} \frac{g^4 a^8}{720Z_g} \text{Tr} F_{ij}^6 - \dots \right). \quad (74)$$

These give the gauge field propagator and a tower of interaction terms. The term with  $2n$  powers of  $F_{ij}$  gives a vertex which attaches to  $2n$  lines, with momenta we will label as  $p_1, p_2, \dots$ , polarization indices we label as  $i_1, i_2, \dots$ , and group indices we label as  $a, b, \dots$ . The Feynman rule for the vertex is that it contributes

$$g^{2n-2} a^{4n-4} (-1)^n \sum_{jk} (ip_{1,j} \delta_{i_1,k} - ip_{1,k} \delta_{i_1,j}) (ip_{2,j} \delta_{i_2,k} - ip_{2,k} \delta_{i_2,j}) \dots \delta(p_1 + p_2 + \dots) \times \\ \times \frac{1}{(2n)!} \left( \text{Tr}(T^a T^b \dots) + \text{all permutations of } a, b, \dots \right). \quad (75)$$

Though the nonrenormalizable terms here have several powers of  $a$ , they lead to diagrams with strong power law UV divergences and are important to the radiative corrections. The first of these is responsible, through diagram (a) in Figure 2, for the first term in Eq. (25).

The first natural question is, is this approximation any good? To answer that, compare the numerical value of the first term in Eq. (25) to the sum of the others. Even with both scalars present, in SU(2) the first term is larger than the others combined by a factor of 3. At large  $N$  the domination is by a factor of 6. Without scalars the domination is twice as large. So it appears that “tadpole” type terms do give the dominant contributions. We should also comment that the contributions from these diagrams are gauge invariant and transverse diagram by diagram.

Now let us compute the two loop “tadpole” type terms. There are insertions of the  $O(a)$  counterterm in the propagator and the vertex of diagram (a), but these cancel. There are also diagrams (b), (c), and (d), which give contributions to  $Z_g^{-1} - 1$  equal to

$$(b) \rightarrow \frac{g^4 a^2 \pi^2}{16\pi^2} \left( \frac{N^4 - 6N^2 + 10}{6N^2} \right) \left( \frac{2 \times 0.3015814}{3} \right), \quad (76)$$

$$(c) \rightarrow \frac{g^4 a^2 \pi^2}{16\pi^2} \left( \frac{(2N^2 - 3)^2}{9N^2} \right) \left( \frac{4}{9} \right), \quad (77)$$

$$(d) \rightarrow -\frac{g^4 a^2 \pi^2}{16\pi^2} \left( \frac{2(5N^4 - 15N^2 - 1)}{15N^2} \right) \left( \frac{1}{9} \right). \quad (78)$$

We have written each contribution as a common “loop counting” type term, a group theoretic term (with powers of  $g^2/4$  taken out into the leading expression), and the product of the symmetry factor and momentum integration. Diagram (c) gives precisely the square of the one loop contribution. For every value of  $N$ , the sum of these 2 loop diagrams is less than but on order the square of the one loop correction.

The numerical constant, 0.3015814, is the integral

$$\int_{[-\pi, \pi]^9} \frac{d^3 k d^3 l d^3 m}{(2\pi)^9} \left( \frac{\tilde{k}_1^2 + \tilde{k}_2^2}{\tilde{k}^2} \right) \left( \frac{\tilde{l}_1^2 + \tilde{l}_2^2}{\tilde{l}^2} \right) \left( \frac{\tilde{m}_1^2 + \tilde{m}_2^2}{\tilde{m}^2} \right) (2\pi)^3 \delta^3(k + l + m), \quad (79)$$

where  $\tilde{k}_i = 2 \sin(k_i/2)$  and  $\tilde{k}^2 = \sum_i \tilde{k}_i^2$ . This integral, like all the integrals arising from the tadpole terms, is completely infrared safe and has no continuum analog.

## B Some particular examples

In this appendix we will plug in numerical values and give step by step instructions for building  $O(a)$  improved Lagrangians for the specific theories of most interest, namely SU(2) plus fundamental Higgs, SU(3) and SU(5) plus adjoint Higgs, and SU(3)×SU(2), each with a fundamental Higgs. The latter is a model for studying the phase transition in the minimal supersymmetric standard model with a light stop squark, which makes including the SU(3) color sector necessary.

In each case the first step is to write down the desired values for  $g^2 a/4$ ,  $x_i = \lambda_i/g^2$ , and  $y_i = m_i^2/g^4$ , which one wants the simulation to correspond to. Then one computes a number of counterterms and uses them to find the “naive” values  $g^2 a_n/4$ ,  $x_{i,n}$ ,  $y_{i,n}$  corresponding to the desired improved values. One constructs the lattice Lagrangian using these, computing two loop mass squared and operator counterterms  $\delta m^2$  and  $\delta \langle \phi^2 \rangle$  exactly as one would in ignorance of the  $O(a)$  improvement scheme, using expressions in [4, 5, 19]. Then one applies  $O(a)$  multiplicative corrections to “naive” measured values of  $(\langle \phi^2 \rangle + \delta \langle \phi^2 \rangle)_{\text{naive}}$  and the surface tension  $\sigma_{\text{naive}}$ .

### B.1 SU(2) + fundamental Higgs

We begin with SU(2) plus fundamental Higgs theory. Writing  $x = \lambda/g^2$  and  $y = m^2/g^4$ , we choose desired values for  $g^2 a/4$ ,  $x$ , and  $y$ , and then compute the counterterms

$$Z_g^{-1} - 1 = 0.6674 \frac{g^2 a}{4}, \quad (80)$$

$$\delta x = (0.01825 - 0.4717x + 0.5839x^2) \frac{g^2 a}{4}, \quad (81)$$

$$Z_m - 1 = (-0.2358 + 0.2919x) \frac{g^2 a}{4}. \quad (82)$$

Now compute the “naive” values

$$g^2 a_n = Z_g g^2 a, \quad (83)$$

$$x_n = Z_g^{-1}(x + \delta x), \quad (84)$$

$$y_n = Z_g^{-2} Z_m y, \quad (85)$$

and construct a lattice lagrangian which, according to naive lattice to continuum relations, should model the continuum theory with these “naive” values. Compute the counterterms  $\delta y_n$  and  $(\delta \langle \phi^\dagger \phi \rangle)_n$  using the two loop expressions in [5], and using  $a_n$ ,  $x_n$ . Do not include the corrections, Eqs. (44,45,46). (Alternately, compute them using  $g^2 a$  and  $x$ , and include Eqs. (44,45,46), but then multiply by  $Z_g^{-2} Z_m$ . The two approaches differ at  $O(g^2 a)$  and it is not clear without the 3 loop calculation which will result in smaller  $O(a)$  additive errors.)

This theory will in fact represent the continuum theory with lattice spacing  $a$ , scalar self-coupling  $\lambda = xg^2$ , and Higgs mass  $m_\phi^2 = yg^4$ , up to  $O(a^2)$  errors (and an  $O(a)$  additive shift to  $y$ ). The  $O(a)$  corrected value for the order parameter is

$$\langle \phi^\dagger \phi \rangle_{\text{corrected}} = Z_g Z_m \left( \langle \phi^\dagger \phi \rangle_{n,\text{meas}} + (\delta \langle \phi^\dagger \phi \rangle)_n \right), \quad (86)$$

and the surface tension is

$$\sigma_{\text{corrected}} = Z_g^2 \sigma_{\text{naive}}. \quad (87)$$

Note that in 3-D units the surface tension is just an inverse area. To convert to 4-D units one multiplies by  $T$ . Physical lengths such as correlation lengths should be corrected by multiplying by  $Z_g^{-1}$ .

## B.2 SU(3) + adjoint Higgs

Suppose we want an improved lattice action for SU(3) plus adjoint Higgs theory at a desired value of  $g^2 a/4$ ,  $x = (\lambda_1 + \lambda_2/2)/g^2$ , and  $y = m_\Phi^2/g^4$ . We compute

$$Z_g^{-1} - 1 = 1.3299 \frac{g^2 a}{4}, \quad (88)$$

$$\delta x = (0.21895 - 1.8867x + 0.7785x^2) \frac{g^2 a}{4}, \quad (89)$$

$$Z_m - 1 = (-0.9434 + 0.4866x) \frac{g^2 a}{4}. \quad (90)$$

Note that the pure gauge (no powers of  $x$ ) corrections are much larger here than in SU(2) plus fundamental Higgs theory, particularly for the self-coupling. This means we need a smaller value of  $g^2 a/4$  to fight down lattice artifacts, especially if we are exploring small  $x$ .

From this point the steps are the same as in SU(2); write down a lattice action which according to “naive” lattice to continuum matching corresponds to the continuum theory at the “naive” values

$$g^2 a_n = A_g g^2 a, \quad (91)$$

$$x_n = Z_g^{-1}(x + \delta x), \quad (92)$$

$$y_n = Z_g^{-2} Z_m y, \quad (93)$$

and compute the counterterms for the mass squared and operator insertion from the equations in [19]. Take data, and correct the  $\text{Tr}\Phi^2$  order parameter by multiplying by  $Z_g Z_m$  and the surface tension by multiplying by  $Z_g^2$ .

## B.3 SU(5) + adjoint Higgs

This case differs from SU(3) plus adjoint Higgs theory only in that there are now two scalar self-couplings  $x_1 = \lambda_1/g^2$  and  $x_2 = \lambda_2/g^2$ , and that there are different numerical values for

the counterterms,

$$Z_g^{-1} - 1 = 2.454 \frac{g^2 a}{4}, \quad (94)$$

$$\delta x_1 = \left( 0.1460 - 3.145x_1 + 1.557x_1^2 + 0.915x_1x_2 + 0.163x_2^2 \right) \frac{g^2 a}{4}, \quad (95)$$

$$\delta x_2 = \left( 0.2433 - 3.145x_2 + 0.584x_1x_2 + 0.311x_2^2 \right) \frac{g^2 a}{4}, \quad (96)$$

$$Z_m - 1 = (-1.572 + 1.265x_1 + 0.457x_2) \frac{g^2 a}{4}. \quad (97)$$

From here on the procedure follows the SU(3) plus adjoint Higgs case. Note that  $g^2 a/4$  must be even smaller for SU(5) than for SU(3), which is not too surprising.

## B.4 SU(3)×SU(2) + fundamental Higgs in each

This theory could be phenomenologically interesting if the left handed stop squark is light enough that a perturbative treatment becomes unreliable. It holds a potentially rich phenomenology, allowing a double phase transition and an exotic color broken symmetric phase [17]. It also nicely illustrates what to do when there are more than one gauge group, and it presents a heirarchy problem which makes its numerical study without  $O(a)$  corrections almost impossible, and its study even with them rather tenuous, for the physical value of  $g_s^2/g_w^2$ .

We will use the notation of Laine and Rajantie. The continuum Lagrangian is

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \text{Tr} G_{ij}^2 + \frac{1}{2} \text{Tr} F_{ij}^2 + (D_i U)^\dagger (D_i U) + (D_i H)^\dagger (D_i H) + \\ & + m_U^2 U^\dagger U + m_H^2 H^\dagger H + \lambda_U (U^\dagger U)^2 + \lambda_H (H^\dagger H)^2 + \gamma U^\dagger U H^\dagger H, \end{aligned} \quad (98)$$

where  $U$ , the squark field, is in the fundamental representation of SU(3) (the field  $G$ , with coupling constant  $g_s$ ) and  $H$ , the light Higgs field, is in the fundamental representation of SU(2) (the field  $F$ , with coupling constant  $g_w$ ).

The theory is characterized by one dimensionful number  $g_s^2$  and 6 dimensionless parameters,  $x_1 = \lambda_U/g_s^2$ ,  $x_2 = \lambda_H/g_s^2$ ,  $x_3 = \gamma/g_s^2$ ,  $y_1 = m_U^2/g_s^4$ ,  $y_2 = m_H^2/g_s^4$ , and  $z = g_w^2/g_s^2$ . Note that we express all of the masses and couplings in terms of  $g_s^2$ . To put the theory on the lattice, we first choose a desired value for each dimensionless number and a value for the lattice spacing in terms of the one dimensionful number, that is we must choose  $g_s^2 a/4$ . The  $O(a)$  improvement is carried out by first computing the counterterms

$$Z_s^{-1} - 1 = 1.2619 \frac{g_s^2 a}{4}, \quad (99)$$

$$Z_w^{-1} - 1 = (0.6674) z \frac{g_s^2 a}{4}, \quad (100)$$

$$\delta x_1 = \left( 0.03514 - 0.8386x_1 + 0.6812x_1^2 + 0.0487x_3^2 \right) \frac{g_s^2 a}{4}, \quad (101)$$

$$\delta x_2 = \left( 0.01825z^2 - 0.4717zx_2 + 0.5839x_2^2 + 0.0730x_3^2 \right) \frac{g_s^2 a}{4}, \quad (102)$$

$$\delta x_3 = (-0.4193x_3 - 0.2358zx_3 + 0.3893x_1x_3 + 0.2919x_2x_3) \frac{g_s^2 a}{4}, \quad (103)$$

$$Z_m - \mathbf{1} = \frac{g_s^2 a}{4} \begin{bmatrix} -0.4193 + 0.3893x_1 & 0.1460x_3 \\ 0.0973x_3 & -0.2358z + 0.2919x_2 \end{bmatrix}, \quad (104)$$

where we choose the convention that  $U^\dagger U$  goes in the upper and  $H^\dagger H$  in the lower row in expressions where  $Z_m$  acts on columns.

Now we implement the  $O(a)$  improvement by defining a set of “naive” paramters,

$$a_n = Z_s a, \quad (105)$$

$$z_n = \frac{Z_w}{Z_s} z, \quad (106)$$

$$x_{1,n} = Z_s(x_1 + \delta x_1), \quad (107)$$

$$x_{2,n} = Z_s(x_2 + \delta x_2), \quad (108)$$

$$x_{3,n} = Z_s(x_3 + \delta x_3), \quad (109)$$

$$\begin{bmatrix} y_{1,n} \\ y_{2,n} \end{bmatrix} = Z_s^{-2} Z_m \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \quad (110)$$

We then use a “naive” continuum to lattice relation to convert these into a lattice action, computing the mass and operator insertion conterterms from the expressions in [19]. Finally, improved values for jumps in order parameters are computed by multiplying the unimproved values by the matrix  $Z_s Z_m$ , the surface tension is improved by multiplying by  $Z_s^2$ , and correlation lengths are rescaled by  $Z_s^{-1}$ .

One point is in order, though. For  $T \simeq 100\text{GeV}$ , the values of the coupling constants are around<sup>8</sup>  $g_s^2/4\pi \simeq 0.087T$  and  $g_w^2/4\pi \simeq 0.032T$ . Hence,  $z \simeq .368 \sim 1/3$ , so the “natural scale” of the strong and weak sectors differ by a factor of 3. To contain  $O(a^2)$  errors in the strong sector, we need  $g_s^2 a/4$  to be fairly small—for instance, we probably want  $Z_s^{-1} - 1$  to be on order 0.1. However, this forces a value of  $(g_w^2 a/4) < 1/30$ , which is a very fine lattice. Yet we must make the lattice volume large enough to contain the rather long correlation lengths of the  $\text{SU}(2)$  sector, which puts some strain on numerical practicality. Hence there may realistically be something of a heirarchy problem involved in studying this system at the physical values for the couplings.

## References

- [1] A. Linde, Phys. Lett. **B 96** (1980) 289.
- [2] P. Ginsparg, Nucl. Phys. **B 170** (1980) 388; T. Applequist and R. Pisarski, Phys. Rev. **D 23** (1981) 2305; S. Nadkarni, Phys. Rev. **D 27** (1983) 917; N. P. Landsman, Nucl. Phys. **B 322** (1989) 498.

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<sup>8</sup>To some readers the value for  $\alpha_s$  may seem low, since the value at the  $Z$  pole is about 0.118. But the 3-D theory value is roughly the value at the  $\overline{\text{MS}}$  renormalization point  $\mu = 7T$ , and  $\alpha_s$  runs quite a bit in between.

- [3] K. Farakos, K. Kajantie, K. Rummukainen, and M. Shaposhnikov, Nucl. Phys. **B 425** (1994) 67.
- [4] K. Farakos, K. Kajantie, K. Rummukainen, and M. Shaposhnikov, Nucl. Phys. **B 442** (1995) 317.
- [5] M. Laine, Nucl. Phys. **B 451** (1995) 484.
- [6] K. Kajantie, M. Laine, K. Rummukainen, and M. Shaposhnikov, Nucl. Phys. **B 466** (1996) 189.
- [7] H. Dosch, J. Kripfganz, A. Laser, and M. Schmidt, Phys. Lett. **B 365** (1996) 213; M. Gurtler, E. Ilgenfritz, J. Kripfganz, H. Perlt, and A. Schiller, Nucl. Phys. **B 483** (1997) 383; M. Gurtler, E. Ilgenfritz, and A. Schiller, hep-lat/9702020; hep-lat/9704013.
- [8] O. Philipsen, M. Teper, and H. Wittig, Nucl. Phys. **B 469** (1996) 445; A. Hart, O. Philipsen, J. Stack, and M. Teper, Phys. Lett. **B 396** (1997) 217.
- [9] G. D. Moore and N. Turok, Phys. Rev. **D 55** (1997) 6538.
- [10] K. Kajantie, M. Laine, K. Rummukainen, and M. Shaposhnikov, Nucl. Phys. **B 493** (1997) 413.
- [11] K. Kajantie, M. Laine, K. Rummukainen, and M. Shaposhnikov, Nucl. Phys. **B 458** (1996) 90.
- [12] J. Cline and K. Kainulainen, Nucl. Phys. **B 482** (1996) 73; hep-ph/9705201; M. Losada, Phys. Rev. **D 56** (1997) 2893; M. Losada and G. Farrar, Phys. Lett. **B 406** (1997) 60; M. Losada, hep-ph/9612337; M. Laine, Nucl. Phys. **B 481** (1996) 43.
- [13] G. D. Moore, Nucl. Phys. **B 493** (1997) 439.
- [14] K. Kajantie, M. Laine, K. Rummukainen, and M. Shaposhnikov, hep-ph/9704416.
- [15] K. Kajantie, M. Laine, J. Peisa, A. Rajantie, K. Rummukainen, and M. Shaposhnikov, hep-ph/9708207.
- [16] M. Karjalainen and J. Peisa, hep-lat/9607023; M. Karjalainen, M. Laine, and J. Peisa, Nucl. Phys. Proc. Suppl. **53** (1997) 475; K. Kajantie, M. Karjalainen, M. Laine, and J. Peisa, cond-mat/9704056; P. Dimopoulos, K. Farakos, and G. Koutsoumbas, hep-lat/9703004.
- [17] D. Bödeker, P. John, M. Laine, and M. Schmidt, Nucl. Phys. **B 497** (1997) 387.
- [18] A. Rajantie, hep-ph/9702255.
- [19] M. Laine and A. Rajantie, hep-lat/9705003.
- [20] K. Symanzik, in Mathematical Problems in Theoretical Physics, eds. R. Schrader et al., Lecture Notes in Physics 153 (Springer, Berlin, 1982); K. Symanzik, Nucl. Phys. **B 226** (1983) 187; 205.

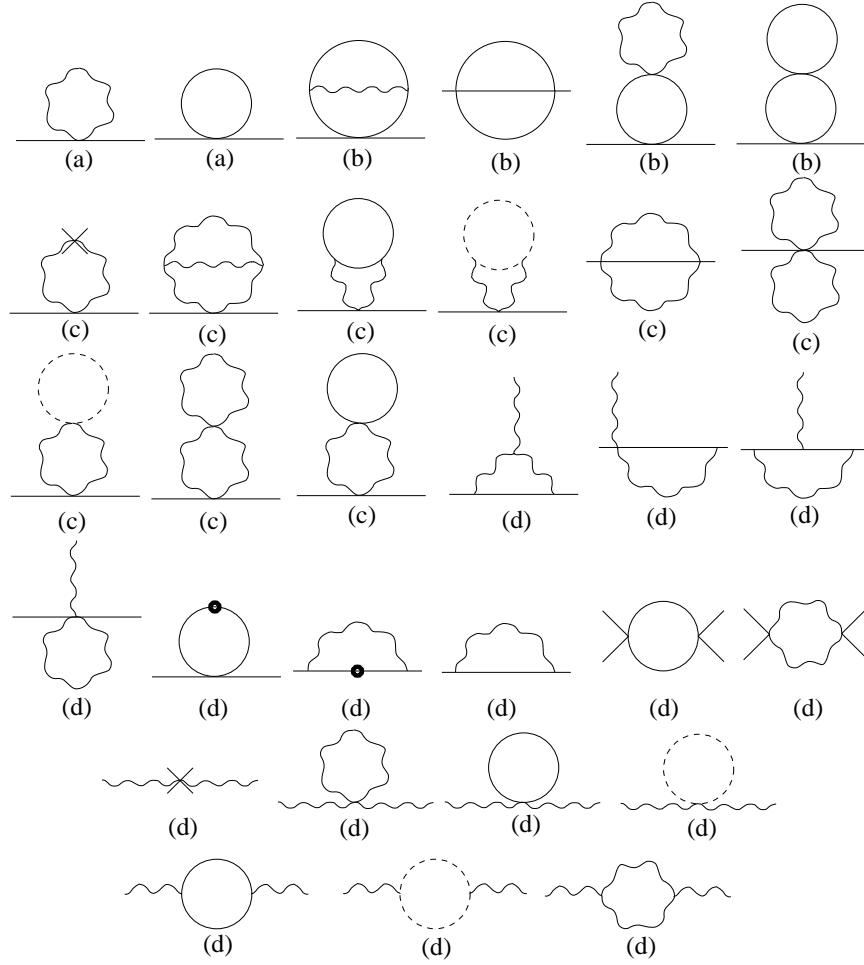


Figure 1: The topologically distinct diagrams needed for the Landau gauge calculation. Wavy lines are gauge propagators, solid lines are scalars, and dotted lines are ghosts. A cross is a self energy correction from the measure and a blot is a  $\phi^2$  operator insertion.

- [21] A. Hasenfratz and P. Hasenfratz, Phys. Lett. **B 93** (1980) 165; R. Dashen and D. Gross, Phys. Rev. **D 23** (1981) 2340; H. Kawai, R. Nakayama, and K. Seo, Nucl. Phys. **B 189** (1981) 40.
- [22] M. Lüscher, S. Sint, R. Sommer, P. Weisz, and U. Wolff, Nucl. Phys. **B 491** (1997) 323.
- [23] H. J. Rothe, “Lattice Gauge Theories: an Introduction” (World Scientific, Singapore, 1992).
- [24] K. Kajantie, K. Rummukainen, and M. Shaposhnikov, Nucl. Phys. **B 407** (1993) 356.

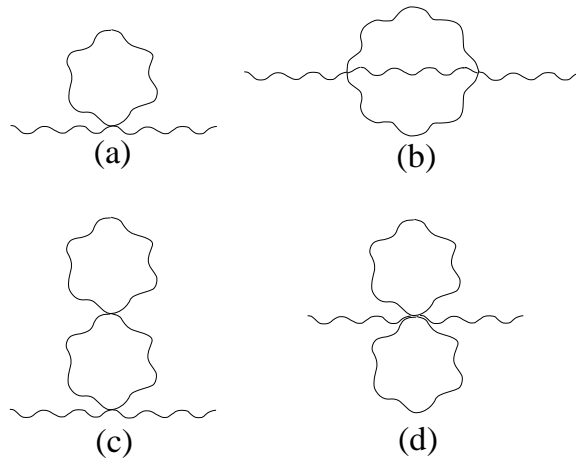


Figure 2: Tadpole type diagrams which correct the gauge field self-energy at one and two loops.